Ex: Compute $\iiint_R e^{z/y}\,dV$ for $R=\{(x,y,z): 1\le y\le 2,\ y\le x\le 2,\ 0\le z\le xy\}$.

Sol:

$$\iiint_R e^{z/y}\,dV$$

$$= \int_{y=1}^{2}\int_{x=y}^{2}\int_{z=0}^{xy} e^{z/y}\,dz\,dx\,dy$$

($u(z)=z/y$, $u(0)=0$; $du=\frac{1}{y}dz$, $u(xy)=x$)

$$= \int_{y=1}^{2}\int_{x=y}^{2}\int_{u=0}^{x} y e^{u}\,du\,dx\,dy$$

$$= \int_{y=1}^{2}\int_{x=y}^{2} y\left[e^{u}\right]_{u=0}^{x}\,dx\,dy$$

$$= \int_{y=1}^{2}\int_{x=y}^{2} y\,(e^{x}-1)\,dx\,dy$$

$$= \int_{y=1}^{2} y\left[e^{x}-x\right]_{x=y}^{2}\,dy$$

$$= \int_{y=1}^{2} y\left((e^{2}-2)-(e^{y}-y)\right)dy$$

$$= \int_{y=1}^{2}\left(y^{2}+(e^{2}-2)y-ye^{y}\right)dy$$

$$= \left[\tfrac{1}{3}y^{3}+\tfrac{1}{2}(e^{2}-2)y^{2}\right]_{y=1}^{2} - \left[\int ye^{y}\,dy\right]_{y=1}^{2}$$

($u=y$, $du=dy$; $dv=e^{y}dy$, $v=e^{y}$)

$$= \tfrac{1}{3}(8-1)+\tfrac{1}{2}(e^{2}-2)(4-1)-\left[ye^{y}-\int e^{y}dy\right]_{y=1}^{2}$$

$$= \tfrac{7}{3}+\tfrac{3}{2}e^{2}-3-\left[ye^{y}-e^{y}\right]_{y=1}^{2}$$

$$= -\tfrac{2}{3}+\tfrac{3}{2}e^{2}-\left((2e^{2}-e^{2})-(e-e)\right)$$

$$= -\tfrac{2}{3}+\tfrac{3}{2}e^{2}-e^{2}$$

$$= \tfrac{1}{2}e^{2}-\tfrac{2}{3}$$